Ordinary Differential Equations.

Higher order Linear Differential Equations.

The general form of a n^{th} order linear differential equation with constant coefficient is $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = R(x)$ — (1)

Where $a_0, a_1, a_2 \dots a_n$ are constants, $a_0 \neq 0$, and $R(x)$ is a function of x.

Here $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^n = \frac{d^n}{dx^n}$

i.e.) $a_0 D^n y + a_1 D^{n-1} y + \dots + a_{n-1} D y + a_n y = R(x)$

$$[a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n] y = R(x)$$

i.e.) $f(D) y = R(x)$

where $f(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$.

Complementary function:

Case (i) All the roots of the A.E are real and distinct.

Then $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$.

Case (2) All the roots of the A.E are real and some are equal.

Then $y = (C_1 + C_2 x + C_3 x^2 + \dots + C_n x^{n-1}) e^{mx}$

~~Case 3)~~

Case 3 ~~The~~ Roots are complex in the form of $\alpha \pm i\beta$.

Then $y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$.

Soln: The given equation is

$$D^2y + Dy - 2y = 0$$

$$(D^2 + D - 2)y = 0.$$

The auxillary equation is (put $D = m$)

$$m^2 + m - 2 = 0$$

$$(m-1)(m+2) = 0$$

$$m = 1, -2$$

∴ The Complementary function is $y = Ae^{m_1 x} + Be^{m_2 x}$

i.e) $y = Ae^{x} + Be^{-2x}$.

(2) Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

Soln: The given equation is

$$(D^3 + 3D^2 + 3D + 2)y = 0$$

The A.E is

$$m^3 + 3m^2 + 3m + 2 = 0$$

$m = -2$ is one root.

By Synthetic division.

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 3 & 2 \\ & 0 & -2 & -2 & -2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

The reduced equation is

$$m^2 + m + 1 = 0.$$

$$\therefore m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore$ The roots are -2, $\frac{-1 \pm \sqrt{3}i}{2}$

The CoF is

$$y = Ae^{-2x} + e^{-\frac{1}{2}x}\left(B\cos\tfrac{\sqrt{3}}{2}x + C\sin\tfrac{\sqrt{3}}{2}x\right)$$

Hence the general solution is

$$y = Ae^{-2x} + e^{-\frac{1}{2}x}\left[B\cos\tfrac{\sqrt{3}}{2}x + C\sin\tfrac{\sqrt{3}}{2}x\right]$$

(3) Solve $y'' + 4y' + 13y = 18e^{-2x}$, $y(0) = 0$, $y'(0) = 4$

Soln: The given equation is

$$(D^2 + 4D + 13)y = 18e^{-2x}$$

The A.E is

$$m^2 + 4m + 13 = 0$$

$$\therefore m = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm 6i}{2} = -2 \pm 3i$$

The C.F is $y = e^{-2x}(C_1\cos 3x + C_2\sin 3x)$

$$PI = \frac{1}{f(D)} \cdot R(x)$$

$$= \frac{1}{D^2+4D+13} 18 e^{-2x}$$ put $D = -2$

$$= \frac{1}{(-2)^2+4(-2)+13} 18 e^{-2x}$$

$$= \frac{1}{4-8+13} 18 e^{-2x}$$

$$= \frac{1}{9} 18 e^{-2x} = 2 e^{-2x}.$$

$\therefore$ The general Solution is

$$y = CF + PI$$

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 2 e^{-2x} \quad \text{(1)}$$

Given $y(0) = 0$.

$\therefore$ put $x = 0$ in (1)

$$y(0) = e^{-2\times 0}(c_1 \cos 0 + c_2 \sin 0) + 2e^{0} = 0$$

$$e^{0}(c_1 + 0) + 2e^{0} = 0$$

$$c_1 + 2 = 0$$

$$\boxed{c_1 = -2}$$

$\cos 0 = 1$
$\sin 0 = 0$

Differentiating (1) wrt. x

$$y'(x) = e^{-2x}(c_1 \times 3 \sin 3x + 3 c_2 \cos 3x)$$

$$+ (c_1 \cos 3x + c_2 \sin 3x)(-2) e^{-2x}$$

$- 4e^{-2x}$

putting $x=0$

$y'(0) = e^0(c_1 \times 3\sin 0 + 3c_2\cos 0)$

$+ (c_1\cos 0 + c_2\sin 0)(-2)e^0 - 4e^0$

$4 = 1\cdot(0 + 3c_2) - 2(c_1 + 0) - 4$

$4 = 3c_2 - 2c_1 - 4$

$4 = 3c_2 - 2\times -2 - 4$

$4 = 3c_2 + 4 - 4$

$4 = 3c_2$

$\boxed{c_2 = 4/3}$

$\therefore y = e^{-2x}(-2\cos 3x + 4/3\sin 3x) + 2e^{-2x}.$

HW: ① Solve $y'' + 4y' + 3y = 8e^{x} + 6e^{2x}$

given $y(0) = 0$ and $y'(0) = 3.$

Ans:

$y = -\frac{21}{10}e^{-3x} + \frac{7}{2}e^{-x} + e^{x} - 2/5\, e^{2x}.$

② $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 2e^{-x},$

$y = -1/5\, e^{2x}(c_1\cos 3x + c_2\sin 2x$

$\frac{\cos 5x}{2}$

$-5^2 + 5D - 6$

$$= \frac{1}{-9+5D-6}\cos 3x - \frac{1}{-25+5D-6}\cos 5x$$

$$= \frac{1}{2}\left[\frac{1}{5D-15}\right]\cos 3x - \frac{1}{2}\left[\frac{1}{5D-31}\right]\cos 5x$$

$$\cancel{= \frac{1}{2}\left\{\frac{5D+15}{5D-15}\right.}$$

$$= \frac{1}{2}\times\frac{1}{5}\left[\frac{1}{D-3}\cos 3x\right] - \frac{1}{2}\left[\frac{1}{5D-31}\right]\cos 5x$$

$$= \frac{1}{10}\left[\frac{D+3}{(D-3)(D+3)}\cos 3x\right] - \frac{1}{2}\left[\frac{5D+31}{(5D-31)(5D+31)}\cos 5x\right]$$

$$= \frac{1}{10}\left[\frac{D+3}{D^2-9}\times\cos 3x\right] - \frac{1}{2}\left[\frac{5D+31}{25D^2-31^2}\right]\cos 5x$$

$$\cancel{= \frac{1}{10}\left[-3\sin 3x + 3\cos 3x\right.}$$

$$= \frac{1}{10}\left[\frac{D+3}{-3^2-9}\cos 3x\right] - \frac{1}{2}\left[\frac{5D+31}{25\times-5^2-31^2}\right]\cos 5x$$

$$= \frac{1}{10}\left[\frac{-3\sin 3x+3\cos 3x}{-18}\right] - \frac{1}{2}\left[\frac{5\times -5\sin 5x+31\cos 5x}{-625-961}\right]$$

$$= \frac{1}{10}\left[\frac{-3\sin 3x+3\cos 3x}{-18}\right] + \frac{1}{2}\left[\frac{-25\sin 5x+31\cos 5x}{1336}\right]$$

$$= \frac{1}{10}\left[\frac{3\sin 3x-3\cos 3x}{18}\right] + \frac{1}{2}\left[\frac{31\cos 5x-25\sin 5x}{1336}\right]$$

$$= \frac{3[\sin 3x-\cos 3x]}{10\cdot 18\ 6} + \frac{31\cos 5x-25\sin 5x}{2672}$$

$$= \frac{\sin 3x - \cos 3x}{60} + \frac{31\cos 5x - 25\sin 5x}{672}$$

Prob. Solve $(D^2 - 4D + 4)y = e^{2x} + \cos 2x$

Given $(D^2 - 4D + 4)y = e^{2x} + \cos 2x$

The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

C.F $y_c = (C_1 x + C_2) e^{2x}$

$$PI_1 = \frac{1}{f(D)} R(x)$$

$$= \frac{1}{D^2 - 4D + 4} e^{2x}$$ Replace D by a $D = 2$

$$= \frac{1}{2^2 - 4\times 2 + 4} e^{2x}$$

$$= \frac{1}{4 - 8 + 4} e^{2x}$$

$$= \frac{1}{0} e^{2x}$$

$$= x \cdot \frac{1}{f'(D)} e^{2x}$$

$$= x \cdot \frac{1}{2D - 4} e^{2x}$$

$$= x \cdot \frac{1}{2\times 2 - 4} e^{2x}$$

$$= x \cdot \frac{1}{4 - 4} e^{2x}$$

$= x \cdot \frac{1}{0} e^{2x}$

$= x^2 \cdot \frac{1}{f''(D)} \cdot e^{2x}$

$= x^2 \cdot \frac{1}{2} e^{2x}$

$PI_1 = \frac{x^2}{2} e^{2x}$

$PI_2 = \frac{1}{D^2 - 4D + 4} \cos 2x$

$= R.P. \frac{1}{D^2 - 4D + 4} e^{i2x}$ Put $D = a$

$= R.P \frac{1}{(2i)^2 - 4 \times 2i + 4} e^{i2x}$

$= R.P \frac{1}{-4 - 8i + 4} e^{i2x}$

$= R.P \frac{1}{-8i} e^{i2x}$

$= R.P \frac{i}{+8} [\cos 2x + i \sin 2x]$

$= R.P \frac{1}{8} [i \cos 2x - \sin 2x]$

$PI_2 = -\frac{1}{8} \sin 2x$

$\therefore$ Solution $y = CF + PI_1 + PI_2$

prob Solve $(D^2+6D+8)y = e^{-2x} + \cos^2 x$

Given $(D^2+6D+8)y = e^{-2x} + \left(\frac{1+\cos 2x}{2}\right)$

A.E $m^2+6m+8 = 0$

$(m+4)(m+2) = 0$

$m = -4, -2$

C.F $Ae^{-2x} + Be^{-4x}$

P.I$_1$ $= \frac{1}{D^2+6D+8} e^{-2x}$

$= \frac{1}{4-12+8} e^{-2x}$ Replace D by -2

$= x \cdot \frac{1}{2D+6} e^{-2x} = x \cdot \frac{1}{-4+6} e^{-2x}$

$= \frac{x}{2} e^{-2x}$

$PI_2 = \frac{1}{D^2+6D+8}\left[\frac{1}{2}e^{0x}\right] = \frac{1}{2} \times \frac{1}{8} e^{0x} = \frac{1}{16}$

$PI_3 = \frac{1}{D^2+6D+8} \cos 2x$

$= R.P \frac{1}{D^2+6D+8} e^{i2x}$

$= R.P \frac{1}{(2i)^2+6(2i)+8} e^{i2x}$

$= R.P \frac{1}{-4+12i+8} e^{i2x}$

$= R.P \frac{1}{4+12i} e^{i2x}$

$= R.P \frac{4-12i}{(4+12i)(4-12i)} e^{i2x}$

$= R.P.\ \frac{4-12i}{16+144}\, e^{i2x}$

$= R.P\ \frac{4-12i}{160}\,(\cos 2x + i\sin 2x)$

$= \frac{1}{160}\left[4\cos 2x + 12\sin 2x\right]$

$= \frac{2\cos 2x + 6\sin 2x}{80}$

Soln is $y = CF + PI_1 + PI_2 + PI_3$

$y = CF + \frac{\cos 2x + 3\sin 2x}{40}$.

$y = Ae^{-2x} + Be^{-4x} + \frac{x}{2}e^{-2x} + \frac{1}{16} + \frac{\cos 2x + 3\sin 2x}{40}$

Solve $(D^2 - 4D + 4)y = e^{2x} + \cos 4x + x^2$

Solution: Given $(D^2 - 4D + 4)y = e^{2x} + \cos 4x + x^2$

The A.E is $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$.

C.F $= y = (Ax + B)e^{2x}$

$PI_1 = \frac{1}{D^2 - 4D + 4} e^{2x}$

$= \frac{1}{4 - 8 + 4} e^{2x}$

$= \frac{1}{0} e^{2x}$ Rule fails.

$= x \cdot \frac{1}{2D - 4} e^{2x}$

$= x \cdot \frac{1}{4-4} e^{2x}$

$= x^2 \cdot \frac{1}{2} e^{2x}$

$PI_1 = \frac{x^2}{2} e^{2x}$

$PI_2 = \frac{1}{D^2-4D+2} \cos 4x$

$= \frac{1}{-16-4D+4} \cos 4x$

$= \frac{1}{-12-4D} \cos 4x$

Omitting higher power

$= -\frac{1}{4} \cdot \frac{1}{D+3} \cos 4x$

$= -\frac{1}{4}\left(\frac{D-3}{(D+3)(D-3)} \cos 4x\right)$

$= -\frac{1}{4}\left[\frac{D-3}{D^2-9} \cos 4x\right]$

$= -\frac{1}{4} \cdot \frac{D-3}{-16-9} \cos 4x$

$= -\frac{1}{4} \frac{D-3}{-25} \cdot \cos 4x$

$= \frac{1}{100} (D-3) \cos 4x$

$= \frac{1}{100} [-\sin 4x (4) - 3 \cos 4x]$

$PI_2 = -\frac{1}{100} [3 \cos 4x + 4 \sin 4x]$

$PI_3 = \frac{1}{D^2-4D+4} x^2$

$= \frac{1}{4[}$

$$= \frac{1}{4\left[1 + \frac{D^2-4D}{4}\right]} x^2$$

$$= \frac{1}{4}\left[1 + \frac{D^2-4D}{4}\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1 - \left(\frac{D^2-4D}{4}\right) + \left(\frac{D^2-4D}{4}\right)^2 - \dots\right] x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + D + \frac{D^4 + 16D^2 - 8D^3}{16} - \dots\right] x^2$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + D^2\right] x^2 \quad \text{Omitting Higher power}$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}(2)\right]$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right]$$

$$PI = PI_1 + PI_2 + PI_3$$

$$= \frac{x^2}{2} e^{2x} + \left(\frac{-1}{100}\right)(3\cos 4x + 4\sin 4x) + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$y = CF + PI$$

$$y = (Ax + B)e^{2x} + \frac{x^2}{2}e^{2x} - \frac{1}{100}(3\cos 4x + 4\sin 4x) + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

problem Based on RHS x^n

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

Given $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

(i.e.) $(D^2 - 5D + 6)y = x^2 + 3$

The A.E. is $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$, $m = 2, 3$

$C.F = Ae^{2x} + Be^{3x}$

$P.I = \frac{1}{D^2 - 5D + 6}(x^2 + 3)$

$= \frac{1}{6\left[1 + \frac{D^2 - 5D}{6}\right]} x^2 + 3$

$= \frac{1}{6}\left[1 + \frac{D^2 - 5D}{6}\right]^{-1}(x^2 + 3)$

$= \frac{1}{6}\left[1 - \left(\frac{D^2 - 5D}{6}\right) + \left(\frac{D^2 - 5D}{6}\right)^2 - \dots\right](x^2 + 3)$

$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25}{36}D^2\right](x^2 + 3)$ omitting D^3 and higher

$= \frac{1}{6}\left[1 + 5/6 + \frac{19}{36}D^2\right](x^2 + 3)$

$= \frac{1}{6}\left[x^2 + 3 + 5/6(2x) + \frac{19}{36}(2)\right]$

$= \frac{1}{6}\left[x^2 + 3 + \frac{5}{3}x + \frac{19}{18}\right]$

$PI = \frac{1}{108}\left[18x^2 + 30x + 73\right]$

$y = CF + PI = Ae^{2x} + Be^{3x} + \frac{1}{108}\left[18x^2 + 30x + 73\right]$

problem Based on RHS $= e^{ax} x$

$$PI = \frac{1}{f(D)} \cdot e^{ax} x = e^{ax} \frac{1}{f(D+a)} x.$$

prob. Find the P.I of $(D^2+1)y = xe^x$

Soln: $P.I = \frac{1}{D^2+1} x e^x = e^x \cdot \frac{1}{(D+1)^2+1} x$

$$= e^x \frac{1}{[D^2+2D+1]+1} x = e^x \frac{1}{D^2+2D+2} x$$

$$= \frac{e^x}{2} \frac{1}{\left[1+\frac{D^2+2D}{2}\right]} x = \frac{e^x}{2}\left[1+\frac{D^2+2D}{2}\right]^{-1} x$$

$$= \frac{e^x}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left(\frac{D^2+2D}{2}\right)^2 - \cdots\right] x$$

$$= \frac{e^x}{2}\left[1-\frac{2D}{2}\right] x \quad \text{Since omitting } D^2$$

$$= \frac{e^x}{2}[1-D]x = \frac{e^x}{2}[x-1]$$

Prob: Find the P.I of $(D-1)^2 y = e^x \sin x$

Given $(D-1)^2 y = e^x \sin x$

$$P.I = \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{[(D+1)-1]^2} \sin x \quad \text{Replace } D \text{ by } D+1$$

$$= e^x \frac{1}{D^2} \sin x$$

$$= e^x \frac{1}{D}[-\cos x] = -e^x \sin x$$

prob: Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

Soln: Given $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

The A.E is $m^2+4m+3=0$

$(m+1)(m+3)=0$

$m=-1,-3$

$C.F = Ae^{-x} + Be^{-3x}$

$$PI_1 = \frac{1}{D^2+4D+3}\left[e^{-x}\sin x\right]$$

$$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x$$

$$= e^{-x}\frac{1}{D^2-2D+1+4D-4+3}\sin x$$

$$= e^{-x}\frac{1}{D^2+2D}\sin x = e^{-x}\frac{1}{-1+2D}\sin x$$

$$= e^{-x}\frac{2D+1}{(2D)^2-1}\sin x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}\sin x$$

$$= e^{-x}\frac{2D+1}{-4-1}\sin x = \frac{-e^{-x}}{5}(2D+1)\sin x$$

$$PI_1 = \frac{-e^{-x}}{5}\left[2\cos x + \sin x\right]$$

$$PI_2 = \frac{1}{D^2+4D+3}xe^{3x}$$

$$= e^{3x}\frac{1}{(D+3)^2+4(D+3)+3}x$$

$$= e^{3x}\frac{1}{D^2+6D+9+4D+12+3}x$$

$$= e^{3x}\frac{1}{D^2+10D+24}x = \frac{e^{3x}}{24}\frac{1}{1+\frac{10}{24}D+\frac{D^2}{24}}x$$

$$= \frac{e^{3x}}{24}\left[1+\frac{5}{12}D+\frac{D^2}{24}\right]^{-1} x$$

$$= \frac{e^{3x}}{24}\left[1-\left(\frac{5}{12}D+\frac{D^2}{24}\right)+\cdots\right] x$$

$$= \frac{e^{3x}}{24}\left[1-\frac{5}{12}D\right] x$$

$$PI_2 = \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

$$y = CF + PI_1 + PI_2$$

$$y = Ae^{-x} + Be^{-3x} - \frac{e^{-x}}{5}\left[2\cos x + \sin x\right] + \frac{e^{3x}}{24}\left[x-\frac{5}{12}\right]$$

Solve $(D^3-1)y = x\sin x$

Given $(D^3-1)y = x\sin x$

A.E is $m^3-1=0$

i.e) $(m-1)(m^2+m+1)=0$

The roots are $m = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$C.F = C_1e^{x} + e^{-x/2}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$PI = \frac{1}{D^3-1}x\sin x$$

$$= x\cdot\frac{1}{D^3-1}\sin x - \frac{3D^2}{(D^3-1)^2}\sin x$$

$$\left[\because \frac{1}{f(D)}xV = x\cdot\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V\right]$$

$$= -x\cdot\frac{1}{D+1}\sin x + \frac{3}{(D+1)^2}\sin x$$

$e^{x} \frac{D-1}{D^2-1} \sin x + \frac{3}{D^2+2D+1} \sin x \Big]$

$= -x \frac{D-1}{-1-1} \sin x + \frac{3}{-1+2D+1} \sin x$

$= \frac{x}{2}\left[D(\sin x) - \sin x\right] + \frac{3}{2}\left[\int \sin x\, dx\right]$

$PI = \frac{x}{2}(\cos x - \sin x) - \frac{3}{2}\cos x$

The general Soln is $y = CF + PI$

$y = c_1 e^{x} + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

$+ \frac{x}{2}(\cos x - \sin x) - \frac{3}{2}\cos x$

H.W Solve $(D^2+5D+4)y = e^{-x}\sin 2x$

Ans : $Ae^{-x} + Be^{-4x} - \frac{e^{-x}}{26}\left[3\cos 2x + 2\sin 2x\right]$

Find the P.I of $(D^2+1)y = xe^{x}$

Soln: $P.I = \frac{1}{D^2+1} xe^{x}$

$= e^{x} \frac{1}{(D+1)^2+1} x \quad [D = D+a]$

$= e^{x} \frac{1}{D^2+2D+1+1} x$

$= e^{x} \frac{1}{D^2+2D+2} x$

$= \frac{e^{x}}{2}\left[\frac{1}{1+\frac{D^2+2D}{2}}\right] x$

$= \frac{e^{x}}{2}\left[1 + \frac{D^2+2D}{2}\right]^{-1} x$

$= \frac{e^{x}}{2}\left[1 - \frac{(D^2+2D)}{2} + \left(\frac{D^2+2D}{2}\right)^2 - \cdots\right] x$

$= \frac{e^x}{2}\left[1 - \frac{2D}{2}\right]x$ omitting the powers

$= \frac{e^x}{2}[1 - D]x$

$= \frac{e^x}{2}[x - D(x)]$

$= \frac{e^x}{2}[x - 1]$

Solve the equation $(D^2+4)y = x^2\cos 2x$

Soln: The A.E is $m^2+4=0$

The roots are $m = \pm i2$

$C.F = A\cos 2x + B\sin 2x$

$P.I = \frac{1}{D^2+4}$ R.P of $x^2 e^{i2x}$

$= R.P\ e^{i2x}\frac{1}{(D+i2)^2+4}x^2$

$= R.P \cdot e^{i2x}\frac{1}{D^2+4iD-4+4}x^2$

$= R.P\ e^{i2x}\frac{1}{4iD\left[1+\frac{D^2}{4iD}\right]}x^2$

$= R.P\ e^{i2x}\frac{1}{4iD}\left[1+\frac{D^2}{4iD}\right]^{-1}x^2$

$= R.P\ \frac{e^{i2x}}{4iD}\left[1+\frac{iD}{4}\right]^{-1}x^2$

$= R.P\left[\frac{e^{i2x}}{4iD}\left[1+\frac{iD}{4} - \frac{D^2}{16} - \frac{iD^3}{64}\right]x^2\right.$

$= R.P$ of $\left(\frac{-i}{4}e^{i2x}\cdot\frac{1}{D}\left[1+\frac{iD}{4} - \frac{D^2}{16} - \frac{iD^3}{64}\right)x^2\right\}$

$$R.p \text{ of } \left\{ \frac{-i}{4} e^{i2x} \frac{1}{D} \left[2x \, dx = -\tfrac{1}{2} \int \tan 2x \, dx \right. \right.$$

$$= Rp \text{ of } \left\{ \frac{-i}{4} e^{i2x} \frac{1}{D} \left[x^2 + \frac{i}{4} 2x - \frac{1}{16} 2 \right] \right\}$$

$$= R.p \text{ of } \left\{ -\frac{i}{4} e^{i2x} \left[\frac{x^3}{3} + \frac{i}{2} \frac{x^2}{2} - \frac{1}{8} x \right] \right\}$$

$$= R.p \text{ of } \left\{ -\frac{i}{4} (\cos 2x + i \sin 2x) \left[\frac{x^3}{3} + \frac{i}{4} x^2 - \frac{1}{8} x \right] \right.$$

$$= R.p \text{ of } \left\{ \left[-\frac{i}{4} \cos 2x + \frac{\sin 2x}{4} \right] \left[\frac{x^3}{3} + \frac{i}{4} x^2 - \frac{1}{8} x \right] \right.$$

~~= R.p of {~~

$$PI = \frac{x^2}{16} \cos 2x + \frac{x^3}{12} \sin 2x - \frac{x \sin 2x}{32}$$

$\therefore$ General Soln is $y = CF + PI$

$$y = A \cos 2x + B \sin 2x + \frac{x^2}{16} \cos 2x + \frac{x^3}{12} \sin 2x - \frac{x}{32} \sin 2x.$$

HW $(D^2 - 3D + 2) y = 2 \cos(2x + 3) + 2e^x$

(1) Ans: $y = c_1 e^x + c_2 e^{2x} - \frac{1}{10} [3 \sin(2x+3) + \cos(2x+3)] - 2xe^x$

(2) $(D^2 + D) y = x^2 + 2x + 4.$

Method of Variation of parameters.

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \quad \text{--- (1)}$$

$$CF = c_1 f_1 + c_2 f_2 \quad \text{--- (2)}$$

c_1, c_2 are Constants & f_1, f_2 are fn of x

Then $PI = P f_1 + Q f_2$ --- (3)

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx. \quad \text{--- (4)}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx \quad (2)$$

$$\therefore y = c_1 f_1 + c_2 f_2 + PI$$

Note: The Wronskian of f_1, f_2 of (1) is gn by $W = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$

(X)

Solve $(D^2+4)y = \sec 2x$ by the method of Variation of parameters

Soln: Given $(D^2+4)y = \sec 2x$

The A.E is $m^2+4=0$

$m = \pm 2i$

$C.F = c_1 \cos 2x + c_2 \sin 2x$

Here $f_1 = \cos 2x$ | $f_2 = \sin 2x$ | $X = \sec 2x$

$f_1' = -2\sin 2x$ | $f_2' = 2\cos 2x$ |

$$f_1 f_2' - f_1' f_2 = 2\cos^2 2x + 2\sin^2 2x$$

$$= 2[\cos^2 2x + \sin^2 2x]$$

$$= 2(1)$$

$$= 2$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2} dx = -\frac{1}{2}\int \tan 2x \, dx$$

$$= -\frac{1}{2}\left[\frac{-\log(\cos 2x)}{2}\right]$$

$$= \frac{1}{4}\log(\cos 2x).$$

$$Q = \int \frac{f_1 x}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos 2x \sec 2x}{2} dx$$

$$= \frac{1}{2}\int \cos 2x \cdot \frac{1}{\cos 2x} dx$$

$$= \frac{1}{2}\int dx = \frac{1}{2}x$$

$$P.I = P f_1 + Q f_2$$

$$= \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x \sin 2x$$

$$y = C.F + P.I$$

$$\therefore y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\log(\cos 2x)\cos 2x + \frac{1}{2}x \sin 2x.$$

Use the method of variation of parameters to solve $(D^2+1)y = \sec x$

Soln: Given $y'' + y = \sec x$

$\Rightarrow (D^2+1)y = \sec x$

The A.E is $m^2+1 = 0$

$m = 0 \pm i$

$C.F = c_1 \cos x + c_2 \sin x$

$= c_1 f_1 + c_2 f_2$

Here $f_1 = \cos x$ | $f_2 = \sin x$ | $X = \sec x$.

$f_1' = -\sin x$ | $f_2' = \cos x$

$$f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x \sec x}{1} dx$$

$$= -\int \frac{\sin x}{\cos x} dx$$

$$= -\int \tan x \, dx$$

$$= \log[\cos x]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \sec x}{1} dx$$

$$= \int dx = x$$

$$PI = P f_1 + Q f_2$$

$$= \log[\cos x] \cos x + x \sin x$$

$$= \cos x \log(\cos x) + x \sin x$$

$$y = C_1 \cos x + C_2 \sin x + \cos x \log(\cos x) + x \sin x$$

Solve $(D^2+a^2)y = \tan ax$ by the method of variation of parameters.

Soln: Given $(D^2+a^2)y = \tan ax$

The A.E is $m^2+a^2=0$

$m = 0 \pm ai$

$$CF = e^{0x}(c_1 \cos ax + c_2 \sin ax)$$

$$CF = c_1 \cos ax + c_2 \sin ax$$

Here $f_1 = \cos ax$ | $f_2 = \sin ax$

$f_1' = -a\sin ax$ | $f_2' = a\cos ax$ | $X = \tan ax$

$$f_1 f_2' - f_2 f_1' = a\cos ax \cos ax - \sin ax(-a\sin ax)$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a[\cos^2 ax + \sin^2 ax]$$

$$= a$$

$$PI = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= -\int \frac{\sin ax \tan ax}{a}\,dx \qquad [\because X = \tan ax$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)\,dx$$

$$= -\frac{1}{a}\left[\frac{1}{a}\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]$$

$$= +\frac{1}{a^2}\left[\sin ax - \log(\sec ax + \tan ax)\right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\, dx$$

$$= \int \frac{\cos ax \tan ax}{a}\, dx$$

$$= \frac{1}{a}\int \sin ax\, dx$$

$$= \frac{1}{a}\left[-\frac{\cos ax}{a}\right]$$

$$\therefore Q = -\frac{1}{a^2}\cos ax$$

$$PI = Pf_1 + Qf_2$$

$$= \frac{1}{a^2}\cos ax\left[\sin ax - \log(\sec ax + \tan ax)\right]$$

$$- \frac{1}{a^2}\sin ax\left[\cos ax\right]$$

$$= \frac{1}{a^2}\cos ax\left[\cancel{\sin ax} - \log(\sec ax + \tan ax) - \cancel{\sin ax}\right]$$

$$= -\frac{1}{a^2}\left[\cos ax \log(\sec ax + \tan ax)\right]$$

$$y = CF + PI$$

$$y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\cos ax \log\left[\sec ax + \tan ax\right]$$

Hw · ① Solve $\frac{d^2y}{dx^2} + y = \tan x$ by the Method of Variation of parameters.

(2) $(D^2+4)y = \tan 2x$

(3) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$.

Cauchy's and Legendre's Linear Equations.

The general form of linear equation of second order is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Homogeneous equations of Euler type (Cauchy's type)

Linear differential Equations with variable Coefficients.

An equation of

Where $a_1, a_2 \ldots a_n$ are constants and $f(x)$ is a fn of x.

eqn (1) can be reduced to linear differential equation with constant coefficients by substituting $x = e^z$ or $z = \log x$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

(or) $x\frac{dy}{dx} = D'y$ where $D' = \frac{d}{dz}$ — (2)

Similarly $x^2\frac{d^2y}{dx^2} = D'(D'-1)y$

$x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$

$x^4\frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y$

Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

~~Sol:~~ Given $(x^2D^2 - xD + 1)y = 0$

put $x = e^z$ | $xD = D'$

$\log x = z$ | $x^2D^2 = D'(D'-1)$

$[D'(D'-1) - D' + 1]y = 0$

$[D'^2 - D' - D' + 1]y = 0$

$(D'^2 - 2D' + 1)y = 0$

A.E $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y = (Az + B)e^z$

$= (A\log x + B)e^{\log x}$

$y = (A\log x + B)x$

Solve $(x^2D^2 + 4xD + 2)y = \log x$ ~~Given that~~

~~Sol:~~

Given $(x^2D^2 + 4xD + 2)y = \log x$

put $x = e^z$

$\log x = z$

$xD = D'$

$x^2D^2 = D'(D'-1)$

$(D'(D'-1) + 4D' + 2)y = z$

$(D'^2 - D' + 4D' + 2)y = z$

$(D'^2 + 3D' + 2)y = z$

A·E is $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -2, -1$

$C \cdot F = Ae^{-2z} + Be^{-z} = A(e^{z})^{-2} + B(e^{z})^{-1}$

$= Ax^{-2} + Bx^{-1} = \frac{A}{x^2} + \frac{B}{x}$

$P \cdot I \quad \frac{1}{D'^2 + 3D' + 2} z$

$= \frac{1}{2\left[1 + \frac{D'^2 + 3D'}{2}\right]} z$

$= \frac{1}{2}\left[1 + \frac{D'^2 + 3D'}{2}\right]^{-1} \cdot z$

$= \frac{1}{2}\left[1 - \left(\frac{D'^2 + 3D'}{2}\right) + \left(\frac{D'^2 + 3D'}{2}\right)^2 - \cdots\right] z$

omitting D'^2

$= \frac{1}{2}\left[z - \frac{3}{2}D'(z)\right] = \frac{1}{2}\left[z - \frac{3}{2}\right]$

$= \frac{1}{4}[2z - 3]$

$= \frac{1}{4}[2\log x - 3]$ — (1)

$y = C \cdot F + PI$

$= \frac{A}{x^2} + \frac{B}{x} + \frac{1}{4}[2\log x - 3]$.

Solve $[x^2D^2 - xD + 1]y = \left(\frac{\log x}{x}\right)^2$

Soln $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

put $x = e^z$ | $xD = D'$

$\log x = z$ | $x^2D^2 = D'(D'-1)$

$\therefore [D'(D'-1) - D' + 1]y = \left(\frac{z}{e^z}\right)^2$

$(D'^2 - D' - D' + 1)y = z^2 e^{-2z}$

$(D'^2 - 2D' + 1)y = z^2 e^{-2z}$

A.E is $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$C.F = (Az + B)e^z = (A\log x + B)e^{\log x}$

$= (A\log x + B)x$

$P.I = \frac{1}{(D'-1)^2} z^2 e^{-2z}$

$= e^{-2z} \frac{1}{[(D'-2)-1]^2} z^2$

$= e^{-2z} \frac{1}{(D'-3)^2} z^2$

$= \frac{e^{-2z}}{9} \cdot \frac{1}{\left(1 - \frac{D'}{3}\right)^2} z^2$

$= \frac{e^{-2z}}{9}\left[1 - \frac{D'}{3}\right]^{-2} z^2$

$= \frac{e^{-2z}}{9}\left[1 + 2\frac{D'}{3} + 3\left(\frac{D'}{3}\right)^2 + \dots\right] z^2$

Omitting D'^3

$= \frac{e^{-2z}}{9}\left[z^2 + \frac{4z}{3} + \frac{6}{9}\right]$

$$= \frac{e^{-2z}}{81}[9z^2+12z+6] = \frac{(e^{z})^{-2}}{81}[9z^2+12z+6]$$

$$= \frac{(x)^{-2}}{81}[9(\log x)^2+12\log x+6]$$

$$= \frac{1}{81x^2}[9(\log x)^2+12(\log x)+6]$$

$$= \frac{1}{27x^2}[3(\log x)^2+4\log x+2]$$

$$y = CF + PI$$

$$\boxed{y = (A\log x + B)x + \frac{1}{27x^2}[3(\log x)^2 + 4(\log x)+2}$$

H.W Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

(1) Ans: $y = \frac{A}{x} + Bx^4 - \frac{1}{6}x^2 - \frac{1}{2}\log x + \frac{3}{8}$

(2) Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Ans: $y = A\log x + B + 2(\log x)^3$

Solve $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

Given $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65\cos(\log x)$

put $x = e^z$, $\log x = z$

So that $xD = D'$ $\quad x^2D^2 = D'(D'-1)$

$x^3D^3 = D'(D'-1)(D'-2)$

$[D'(D'-1)(D'-2) + 3D'(D'-1) + D' + 8]y = 65\cos z$

$[D'^3 - \cancel{3D'^2} + \cancel{2D'} + \cancel{3D'^2} - \cancel{2D'} + 8]y = 65\cos z$

$[D'^3 + 8]y = 65\cos z$

A.E is $m^3+8=0$

$m=-2$ is a root

$$\begin{array}{c|cccc} -2 & 1 & 0 & 0 & 8 \\ & 0 & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

$m^2-2m+4=0$

$m = \dfrac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}\, i$

$C.F = c_1 e^{-2z} + e^z[c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z]$

$= c_1 (e^z)^{-2} + e^z[c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z]$

$= c_1 x^{-2} + e^{\log x}[c_2 \cos\sqrt{3}\log x + c_3 \sin\sqrt{3}\log x$

$P.I = \dfrac{1}{D'^3+8} \cdot 65 \cos z$

$= 65 \dfrac{1}{-D'+8} \cos z \qquad D'^2 \text{ by } -1^2$

$= 65 \dfrac{1}{8-D'} \cos z$

$= 65 \dfrac{8+D'}{(8-D')(8+D')} \cos z$

$= 65 \dfrac{8+D'}{64-D'^2} \cos z$

$= 65 \dfrac{8+D'}{64+1} \cos z \qquad D'^2 = -1^2 = -1$

$= \cancel{65} \dfrac{8+D'}{\cancel{65}} \cos z$

$= 8\cos z - \sin z$

$= 8\cos(\log x - \sin(\log x)$

$y = c_1 x^{-2} + x[c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)$
$+ 8\cos(\log x) - \sin(\log x)$

Legendre's Linear D.E

An equation of the form:

$$(ax+b)^n \frac{d^n y}{dx^n} + C_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + C_n y = 0$$

putting $ax+b = e^z$

$$z = \log(ax+b)$$

$$D' = \frac{d}{dz}, \quad (ax+b) D = aD'$$

$$(ax+b)^2 D^2 = a^2 D'(D'-1)$$

Solve $(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4$

Soln: Given $[(x+2)^2 D^2 - (x+2)D + 1] y = 3x+4$

put $x+2 = e^z \Rightarrow x = e^z - 2$

$$\log(x+2) = z$$

$$(x+2) D = 1 \cdot D'$$

$$(x+2)^2 D^2 = D'(D'-1)$$

$$[D'(D'-1) - D'+1] y = 3(e^z - 2) + 4$$

$$[D'^2 - D' - D' + 1] y = 3e^z - 6 + 4$$

$$[D'^2 - 2D' + 1] y = 3e^z - 2$$

The A.E $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = (Az + B) e^z = [A \log(x+2) + B](x+2)$$

$$PI_1 = \frac{1}{(D'-1)^2} 3e^z$$

Replace

$= 3 \cdot \frac{1}{(1-1)^2} e^z$ $D' = 1$

$= z \cdot \frac{1}{2(D'-1)} 3e^z$

$= \frac{z}{2} \cdot \frac{1}{1-1} 3e^z$ Replace D' by 1

$= z^2 \left(\frac{1}{2}\right) 3e^z = \frac{z^2}{2} 3e^z = \frac{3}{2} z^2 e^z$

$= \frac{3}{2} [\log(x+2)]^2 (x+2)$

$PI_2 = \frac{1}{(D'-1)^2} [-2e^{0z}] = -2 \frac{1}{(-1)^2} = -2$

$y = CF + PI_1 + PI_2$

$y = [A \log(x+2) + B](x+2) + \frac{3}{2} [\log(x+2)]^2 (x+2) - 2$

Solve $[(x+1)^2 D^2 + (x+1)D + 1]y = 4\cos[\log(x+1)]$

Soln: $[(x+1)^2 D^2 + (x+1)D + 1]y = 4\cos[\log(x+1)]$

put $1+x = e^z$

$z = \log(1+x)$

$(x+1)D = D'$

$(x+1)^2 D^2 = D'(D'-1)$

$[D'(D'-1) + D' + 1]y = 4[\cos z]$

A E is $m^2 + 1 = 0$

$m = \pm i$

$CF = A\cos z + B\sin z$

$= A\cos[\log(x+1)] + B\sin[\log(x+1)]$

$$P.I = \frac{1}{D'^2+1} 4\cos z$$

$$= 4 \cdot \frac{1}{D'^2+1} \cos z$$

$$= 4 \cdot \frac{1}{-1+1} \cos z$$ Replace D'^2 by -1^2

$$= 4 \cdot \frac{1}{0} \cos z$$

$$= 4z \cdot \frac{1}{2D'} \cos z = 2z \cdot \frac{1}{D'} \cos z$$

$$= 2z \int \cos z \, dz$$

$$= 2z \sin z$$

$$= 2\log(1+x) \sin[\log(1+x)]$$

$\therefore y = CF + PI$

$$y = A\cos[\log(x+1)] + B\sin[\log(x+1)] + 2\log(x+1) \times \sin[\log(x+1)]$$

Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

Soln. Given

$$[(3x+2)^2 D^2 + 3(3x+2)D - 36]y = 3x^2+4x+1$$

put $3x+2 = e^z$

$$\log(3x+2) = z$$

$$3x = e^z - 2$$

$$x = \frac{1}{3}e^z - \frac{2}{3}$$

Let $(3x+2)D = 3D'$

$$(3x+2)^2 D^2 = 9D'(D'-1)$$

$$[9D'(D'-1) + 3(3D') - 36]y = 3\left[\frac{1}{3}e^z - \frac{2}{3}\right]^2 + 4\left[\frac{1}{3}e^z - \frac{2}{3}\right] + 1$$

$$[9D'^2 - 9D' + 9D' - 36]y = 3\left[\frac{1}{9}e^{2z} + \frac{4}{9} - \frac{4}{9}e^{z}\right]$$

$$+\frac{4}{3}e^{z} - \frac{8}{3} + 1$$

$$[9D'^2 - 36]y = \frac{1}{3}e^{2z} + \frac{4}{3} - \frac{4}{3}e^{z} + \frac{4}{3}e^{z} - \frac{8}{3}+1$$

$$= \frac{1}{3}e^{2z} - \frac{4}{3} + 1 = \frac{1}{3}e^{2z} - \frac{1}{3}$$

$\div 9, \Rightarrow (D'^2 - 4)y = \frac{1}{27}e^{2z} - \frac{1}{27}$

A.E $m^2 - 4 = 0$

$m^2 = 4$

$m = \pm 2$

$$C.F = Ae^{2z} + Be^{-2z} = A(e^{z})^2 + B(e^{z})^{-2}$$

$$= A(3x+2)^2 + B(3x+2)^{-2}$$

$$P.I_1 = \frac{1}{D'^2 - 4} \frac{e^{2z}}{27}$$

$$= \frac{1}{27} \cdot \frac{1}{4-4} e^{2z} \quad \text{Replace } D' \text{ by } 2$$

$$= \frac{1}{27} \cdot z \frac{1}{2D'} e^{2z}$$

$$= \frac{1}{54} z \frac{1}{D'} e^{2z}$$

$$= \frac{z}{54} \cdot \frac{e^{2z}}{2} = \frac{z}{108} e^{2z}$$

$$= \frac{z}{108}(e^{z})^2 = \frac{z}{108}(3x+2)^2$$

$$= \frac{\log(3x+2)}{108}(3x+2)^2$$

$$PI_2 = \frac{1}{D'^2 - 4} \frac{e^{0z}}{27}$$

$$= \frac{1}{27} \cdot \left(\frac{1}{-4}\right) e^{0z} = -\frac{1}{108}$$

$y = CF + PI_1 + PI_2$

$$y = A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}(3x+2)^2 \log(3x+2)$$

$$+ \frac{1}{108}.$$

HW Solve $(2x+3)\cdot\frac{d^2y}{dx^2} - (2x+3)\cdot\frac{dy}{dx} - 12y = 6x$

Ans.

$$y = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4}$$

Simultaneous First order Linear Equations with Constant Coefficients.

Solve $\frac{dx}{dt} + 2x - 3y = t$; $\frac{dy}{dt} - 3x + 2y = e^{2t}$

Soln: Given $\frac{dx}{dt} + 2x - 3y = t$ — (A)

$$Dx + 2x - 3y = t$$

$$(D+2)x - 3y = t. \quad \text{— (1)}$$

$$\frac{dy}{dt} - 3x + 2y = e^{2t} \quad \text{— (B)}$$

$$Dy - 3x + 2y = e^{2t}$$

$$(D+2)y - 3x = e^{2t}.$$

$$-3x + (D+2)y = e^{2t} \quad \text{--- (2)}$$

$$(1) \times 3 \Rightarrow 3(D+2)x - 9y = 3t$$

$$(2) \times (D+2) \Rightarrow -3(D+2)x + (D+2)^2 y = (D+2)e^{2t}$$

$$-9y + (D+2)^2 y = 3t + (D+2)e^{2t}$$

$$-9y + (D^2+4D+4)y = 3t + 2e^{2t} + 2e^{2t}$$

$$(D^2+4D-5)y = 3t + 4e^{2t}$$

A.E $m^2+4m-5=0$

$$(m+5)(m-1)=0$$

$$m = -5, 1$$

$$C.F = Ae^{t} + Be^{-5t}$$

$$P.I_1 = \frac{1}{D^2+4D-5}(3t)$$

$$= 3 \frac{1}{(-5)\left[1-\left(\frac{D^2+4D}{5}\right)\right]} t$$

$$= -\frac{3}{5}\left[1-\left(\frac{D^2+4D}{5}\right)\right]^{-1} t$$

$$= -\frac{3}{5}\left[1+\frac{D^2+4D}{5}+\left(\frac{D^2+4D}{5}\right)^2+\dots\right](t)$$

$$= -\frac{3}{5}\left[t+\frac{4}{5}(D(t))\right]$$

$$= -\frac{3}{5}\left[t+\frac{4}{5}\right]$$

$$= -\frac{3}{5}t - \frac{12}{25}$$

$$PI_2 = \frac{1}{D^2+4D-5} 4e^{2t}$$

$$= 4 \cdot \frac{1}{4+8-5} e^{2t} = 4 \times \tfrac{1}{7} e^{2t} = \tfrac{4}{7} e^{2t}$$

$$y = CF + PI_1 + PI_2$$

$$= Ae^{t} + Be^{-5t} + \left[-\frac{3}{5}t - \frac{12}{25}\right] + \frac{4}{7}e^{2t}$$

$$\boxed{y = Ae^{t} + Be^{-5t} - \tfrac{3}{5}t - \frac{12}{25} + \tfrac{4}{7}e^{2t}}$$

$$Dy = \frac{dy}{dt} = Ae^{t} - 5Be^{-5t} - \tfrac{3}{5} + \tfrac{8}{7}e^{2t}$$

$$(3) \Rightarrow 3x = \frac{dy}{dt} + 2y - e^{2t}$$

$$3x = \left[Ae^{t} - 5Be^{-5t} - \tfrac{3}{5} + \tfrac{8}{7}e^{2x}\right]$$

$$+ 2\left[Ae^{t} + Be^{-5t} - \tfrac{3}{5}t - \frac{12}{25} + \tfrac{4}{7}e^{2t}\right] - e^{2t}$$

$$= Ae^{t} - 5Be^{-5t} - \tfrac{3}{5} + \tfrac{8}{7}e^{2t} + 2Ae^{t} + 2Be^{-5t}$$

$$- \tfrac{6}{5}t - \frac{24}{25} + \tfrac{8}{7}e^{2t} - e^{2t}$$

$$= 3Ae^{t} - 3Be^{-5t} - \tfrac{6}{5}t - \frac{39}{25} + \tfrac{9}{7}e^{2t}$$

$$x = Ae^{t} - Be^{-5t} - \tfrac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

Hence the desired Soln are

$$x = Ae^{t} - Be^{-5t} - \tfrac{2}{5}t + \tfrac{3}{7}e^{2t} - \frac{13}{25}$$

$$y = Ae^{t} + Be^{-5t} - \tfrac{3}{5}t + \tfrac{4}{7}e^{2t} - \frac{12}{25}$$

29/8/23.